Solving Polynomial Equations

Introduction

Linear and quadratic equations, dealt within Blocks 1 and 2 are members of a class of equations called polynomial equations. These have the general form:

$$a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0 = 0$$

in which x is a variable and a_n, a_n−1,...,a_2, a_1, a0 are given constants. Also n must be a positive integer. Examples include $x^3 + 7x^2 + 3x-2=0$, $5x^4 -7x^2 = 0$ and $-x^6 +x^5 -x^4 = 0$. In this block you will learn how to factories some polynomial expressions and solve some polynomial equations.

1. Multiplying polynomials together

Key Point

A polynomial expression is one of the form

$$a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$$

where a_0, a_1, ... , a_n are known coefficients (numbers) and x is a variable.
n must be a positive integer.

For example $x^3 - 17x^2 + 54x - 8$ is a polynomial expression in x. The polynomial may be expressed in terms of a variable other than x. So, the following are also polynomial expressions:

$$t^3 - t^2 + t - 3 \quad z^5 - 1 \quad w^4 + 10w^2 - 12$$

Note that only non-negative whole number powers of the variable x are allowed in a polynomial expression. In this block you will learn how to factorise simple polynomial expressions and how to solve some polynomial equations. You will also learn the technique of equating coefficients. This process is very important when we need to perform calculations involving partial fractions which will be considered in Block 6.
The degree of a polynomial is the highest power to which the variable is raised. Thus x^3+6x+2 has degree 3, $t^6 - 6t^4 + 2t$ has degree 6, and $5x + 2$ has degree 1.
Let us consider what happens when two polynomials are multiplied together. For example

$$(x + 1)(3x - 2)$$

is the product of two first degree polynomials. Expanding the brackets we obtain

$$(x + 1)(3x - 2) = 3x^2 + x - 2$$

Solving Polynomial Equations

which is a second degree polynomial.
In general we can regard a second degree polynomial, or quadratic, as the product of two first degree polynomials, provided that the quadratic can be factorized.
On the other hand

$$(x - 1)(x^2 + 3x - 7) = x^3 + 2x^2 - 10x + 7$$

is a third degree, or cubic, polynomial which is thus the product of a linear polynomial and a quadratic polynomial. In general we can regard a cubic polynomial as the product of a linear polynomial and a quadratic polynomial. This fact will be important in the following section when we come to factorise cubics.

Key Point

A cubic expression is a linear expression times a quadratic expression

2. factorisation polynomial expressions and equating coefficients

Eventually we will consider how we might find the solution to some simple polynomial equations. An important part of this process is being able to express a complicated polynomial into a product of simpler polynomials. This involves factorisation. Factorisation of polynomial expressions can sometimes be achieved if one or more of the factors is already known. This requires a knowledge of the technique of 'equating coefficients' which is illustrated in the following example.

Example factorise the expression $x^3 - 17x^2 + 54x - 8$ given that one of the factors is $(x - 4)$.

Solution
Given that $x - 4$ is a factor we can write

$$x^3 - 17x^2 + 54x - 8 = (x - 4) \times (\text{a quadratic polynomial})$$

The polynomial must be quadratic because the expression on the left is cubic. Suppose we write this quadratic as ax2 + bx + c where a, b and c are unknown numbers which we will try to find. Then

$$x^3 - 17x^2 + 54x - 8 = (x - 4)(ax^2 + bx + c)$$

Removing the brackets on the right and collecting like terms together we have

$$x^3 - 17x^2 + 54x - 8 = ax^3 + (b - 4a)x^2 + (c - 4b)x - 4c$$

Equating coefficients means that we compare the coefficients of each term on the left with the corresponding term on the right. Thus if we look at the x^3 terms on each side we see that

Solving Polynomial Equations

$$x^3 = ax^3$$

that is a must equal 1. Similarly by equating coefficients of x^2 we find

$$-17 = b - 4a$$

With a = 1 we have $-17 = b - 4$ so that b must equal -13. Finally, equating constant terms we find

$$-8 = -4c$$

so that c = 2.

Solution
Check for yourself that with these values of c and b, the coefficient of x is the same on both sides. We can now write the polynomial expression as

$$x^3 - 17x^2 + 54x - 8 = (x - 4)(x^2 - 13x + 2)$$

More exercises for you to try

Factorise the given polynomial expressions

1. $x^3 - 6x^2 + 11x - 6$, given that $x - 1$ is a factor
2. $x^3 - 7x - 6$, given that $x + 2$ is a factor
3. $2x^3 + 7x^2 + 7x + 2$, given that $x + 1$ is a factor
4. $3x^3 + 7x^2 - 22x - 8$, given that $x + 4$ is a factor

3. Polynomial equations

When a polynomial expression is equated to zero, a polynomial equation is obtained. Linear and quadratic equations, which you have already met, are particular types of polynomial equation.

Key Point

A polynomial equation has the form

$$a_n x^n + a_{n-1} x^{n-1} + \ldots a_2 x^2 + a_1 x + a_0 = 0$$

where $a_0, a_1, \ldots, a_n$ are known coefficients, and x represents an unknown whose value(s) are to be found.

Solving Polynomial Equations

Polynomial equations of low degree have special names. A polynomial equation of degree 1 is a linear equation and such equations have been solved in Block 1. Degree 2 polynomials are quadratics; degree 3 polynomials are called cubics; degree 4 equations are called quartics and so on. The following are examples of polynomial equations:

$$5x^6 - 3x^4 + x^2 + 7 = 0,\ -7x^4 + x^2 + 9 = 0,\ t^3 - t + 5 = 0,\ w^7 - 3w - 1 = 0$$

Recall that the degree of the equation is the highest power of x occurring. The solutions or roots of the equation are those values of x which satisfy the equation.

Key Point

A polynomial equation of degree n has n roots

Some (possibly all) of the roots may be repeated
Some (possibly all) of the roots may be complex.

Example Verify that $x = -1$, $x = 1$ and $x = 0$ are solutions (roots) of the equation

$$x^3 - x = 0$$

Solution
We substitute each value in turn into $x^3 - x$.

$$(-1)^3 - (-1) = -1+1\ 0$$

so $x = -1$ is clearly a root. It is easy to verify similarly that $x = 1$ and $x = 0$ are also solutions.

We now consider ways in which polynomial equations of higher degree can be solved. Exercises Verify that the given values are solutions of the given equations.
1. $x^2 - 5x + 6$, $x = 3$, $x = 2$
2. $2t^3 + t\ 2 - t$, $t = 0$, $t = -1$, $t = 1/2$.

4. Solving polynomial equations when one solution is known

In Block 2 we gave a formula which can be used to solve quadratic equations. Unfortunately when dealing with equations of higher degree no simple formulae exist. If one of the roots can be spotted we can sometimes find the others by the method shown in the next example.

Example Let the polynomial expression $x^3 - 17x^2 + 54x - 18$ be denoted by P(x). Verify that $x = 4$ is a solution of the equation $P(x) = 0$. Hence find the other solutions.

Solving Polynomial Equations

Solution

We substitute x = 4 into the polynomial expression P(x):

$$4^3 - 17(4^2) + 54(4) - 8 = 64 - 272 + 216 - 8 = 0$$

So, when x = 4 the left-hand side equals zero. Hence x = 4 is indeed a solution. Knowing that x = 4 is a root we can state that (x − 4) must be a factor of P(x). Therefore P(x) can be re-written as a product of a linear and a quadratic term:

$$P(x) = x^3 - 17x^2 + 54x - 8 = (x - 4) \times (\text{quadratic polynomial})$$

The quadratic polynomial has already been found on page 4. So the given equation can be written

$$P(x) = x^3 - 17x^2 + 54x - 8 = (x - 4)(x^2 - 13x + 2) = 0$$

In this form we see that

$$x - 4 = 0 \text{ or } x^2 - 13x + 2 = 0$$

Solution

The first equation gives x = 4 which we already knew. The second must be solved using one of the methods for solving quadratic equations given in Block 2. For example, using the formula we find

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \text{ with } a = 1, b = -13, c = 2$$

$$= \frac{13 \pm \sqrt{(-13)^2 - 4.1.2}}{2}$$

$$= \frac{13 \pm 161}{2}$$

$$\frac{13 \pm 12.6886}{2}$$

So x = 12.844 and x = 0.156 are roots of $x^2 - 13x + 2$. Hence the solutions of P(x) = 0 are x = 4, x = 12.844 and x = 0.156.