Fluid Statics and Dynamics

Thrust and Pressure

A perfect fluid resists force normal to its surface and offers no resistance to force acting tangential to it surface. A heavy log of wood can be drawn along the surface of water with very little effort because the force applied on the log of wood is horizontal and parallel to the surface of water. Thus fluids are capable of exerting normal stress on the surface with it is in contact Force exerted perpendicular to a surface is called thrust and thrust per unit area is called pressure

Variation of pressure with height

Let h be the height of the liquid column in a cylinder of cross sectional area A. If ρ is the density of the liquid, then weight of the liquid column W is given by W = mass of liquid column × g = Ahρg By definition, pressure is the force acting per unit area.

$$Pressure = weight\ of\ liquid\ column\ /area\ of\ cross - \text{section}$$

$$P = Ah\rho g/\ A = h\rho g$$

$$dP = \rho g\ (dh)$$

This differential relation shows that the pressure in a fluid increases with depth or decreases with increased elevation. Above equation holds for both liquids and gases . Liquids are generally treated as incompressible and we may consider their density ρ constant for every part of liquid. With ρ as constant, equation may be integrated as it stands, and the result is

$P = P_0 + \rho gh$

The pressure P_0 is the pressure at the surface of the liquid where h = 0

Force due to fluid on a plane submerged surface

The pressure at different points on the submerged surface varies so to calculate the resultant force, we divide the surface into a number of elementary areas and we calculate the force on it first by treating pressure as constant then we integrate it to get the net

force i.e $F_R = \int P\ (dA)$

The point of application of resultant force must be such that the moment of the resultant force about any axis is equal to the moment of the distributed force about the axis

Solved Numerical

Q) Water is filled upto the top in a rectangular tank of square cross-section. The sides of cross-section is a and height of the tank is H. If density of water is ρ, find force on the bottom of the tank and on one of its wall. Also calculate the position of the point of application of the force on the wall

Solution

Force on the bottom of thank

Area of bottom of tank = a^2

Force = pressure × Area Force = $H\rho g a^2$

Force on the wall and its point of application

Force on the wall of the tank is different at different heights so consider a segment at depth h of thickness dh

Pressure at depth h = hρg

Area of strip = a dh

Force on strip dF = hρg a dh

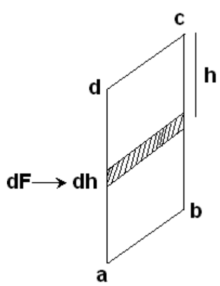
Total force at on the wall

$$F = \int_0^H \rho g a h\ dh$$

$$F = \rho g a \left[h^2/2 \right]_0^H$$

$$F = \rho g a H^2/2$$

The point of application of the force on the wall can be calculated by equating the moment of resultant force about any line,



say dc to the moment of distributioed force about the same line dc Moment of dF about line cd = dF (h) = (hρgadh) h = $\rho g a h^2 dh$

∴ Net moment of distributed forces

$\rho g a \int_0^H h^2 dh = \rho g a H^3/3$ Let the point of application of the net force is at a depth ‘x’ from the line cd Then the torque of the resultant force about the line cd = $Fx = \rho g a H^2/2\, x$

Now Net moment of distribution of force = Torque

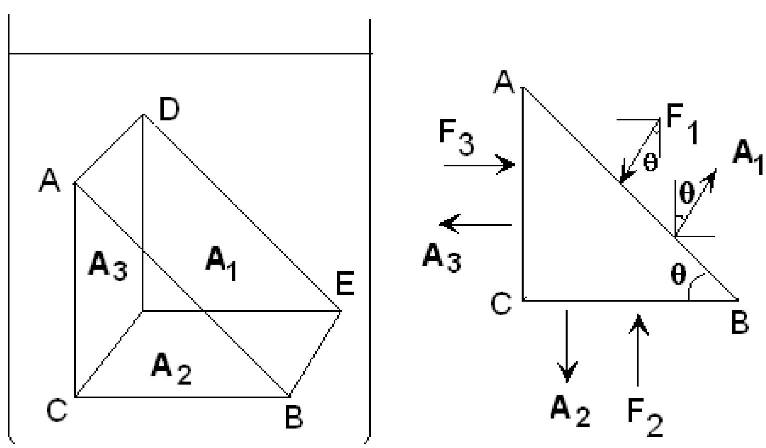
$$\rho g a H^3/3 = \rho g a H^2/2\, x$$

$$x = 2H/3$$

Hence, the resultant force on the vertical wall of the tank will act at a depth 2H/3 from the free surface of water or at the height of H/3 from bottom of tank

Pascal’s Law

Pascal’s law states that if the effect of gravity can be neglected then the pressure in an incompressible fluid in equilibrium is the same everywhere.. This statement can be verified as follows Consider a small element of liquid in the interior of the liquid at rest. The liquid element is in the shape of prism consisting of two right angled triangle surfaces



Let the areas of surface ADEB, CFEB, ADFC be A_1, A_2, A_3 It is clear from figure that

$A_2 = A_1\cos\theta$ and $A_3 = A_1\sin\theta$

Also, since liquid element is in equilibrium $F_3 = F_1\cos\theta$ and $F_3 = F_1\sin\theta$

now pressure on surface ADEB is $P_1 = F_1 / A_1$

Pressure on the surface CFED is

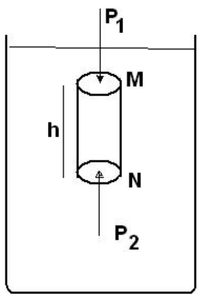
$$P_2 = F_2/A_2 = F_1\cos\theta/A_1\cos\theta = F_1\, A_1$$

And pressure on the surface ADFC is

$$P_3 = F_3\, A_3 = F_1 sin\theta / A1 sin\theta = F_1\, A_1$$

So, $P_1 = P_2 = P_3$
Since θ is arbitrary this result holds for any surface. Thus Pascal's law is verifiedPascal's

law and effect of gravity



When gravity is taken into account, Pascal's law is to be modified. Consider a cylindrical liquid column of height h and density ρ in a vessel as shown in the Fig.
If the effect of gravity is neglected, then pressure at M will be equal to pressure at N.
But, if force due to gravity is taken into account, then they are not equal.
As the liquid column is in equilibrium, the forces acting on it are balanced. The vertical forces acting are

(i) Force P_1A acting vertically down on the top surface.
(i) Weight mg of the liquid column acting vertically downwards.
(ii) Force P_2A at the bottom surface acting vertically upwards. where P_1 and P_2 are the pressures at the top and bottom faces, A is the area of cross section of the circular face and m is the mass of the cylindrical liquid column.

At equilibrium, $P_1A + mg - P_2A = 0$ or $P_1A + mg = P_2A$

$P_2 = P_1 + mg\ A$ But $m = Ah\rho$

$\therefore P_2 = P_1 + Ah\rho gA$

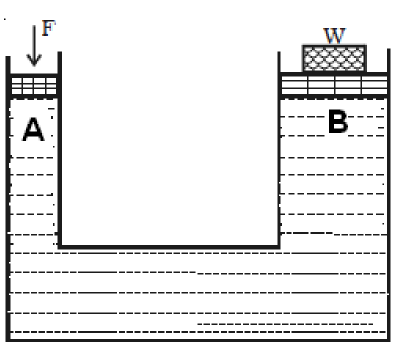
(i.e) $P_2 = P_1 + h\rho g$

This equation proves that the pressure is the same at all points at the same depth. This results in another statement of Pascal's law which can be stated as change in pressure at any point in an enclosed fluid at rest is transmitted undiminished to all points in the fluid and act in all directions.

Characteristics of the fluid pressure

(i) Pressure at a point acts equally in all directions
(ii) Liquids at rest exerts lateral pressure, which increases with depth
(iii) Pressure acts normally on any area in whatever orientation the area may be held
(iv) Free surface of a liquid at rest remains horizontal
(v) pressure at every point in the same horizontal line is the same inside a liquid at rest
(vi) liquid at rest stands at the same height in communicating vessels

Application of Pascal's law



An important application of Pascal's law is the hydraulic lift used to lift heavy objects. A schematic diagram of a hydraulic lift is shown in the Fig.. It consists of a liquid container which has pistons fitted into the small and large opening cylinders. If a_1 and a_2 are the areas of the pistons A and B respectively, F is the force applied on A and W is the load on B, then

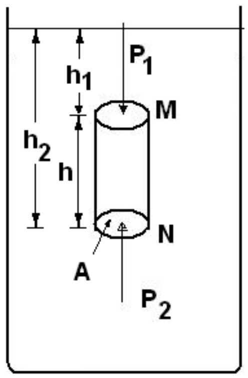
$$F/a_1 = W/a_2$$

$$F = W\, a_1 / \, a_2$$

This is the load that can be lifted by applying a force F on A. In the above equation a_2/a_1 is called mechanical advantage of the hydraulic lift. One can see such a lift in many automobile service stations.

Buoyancy and Archimedes principle

If an object is immersed in or floating on the surface of a liquid, it experiences a net vertically upward force due to liquid pressure. This force is called as Buoyant force or force of Buoyancy and it acts from the centre of gravity of the displaced liquid. According to Archimedes principle, "the magnitude of force of buoyancy is equal to the weight of the displaced liquid" To prove Archimedes principle, consider a body totally immersed in a liquid as shown in the figure. The vertical force on the body due to liquid pressure may be found most easily by considering a cylindrical volume similar to that one shown in figure



The net vertical force on the element is
$dF = (P_2 - P_1)\, A$

$$F = [(P_0 + h_2 \rho g) - (P_0 + h_1 \rho g)]$$

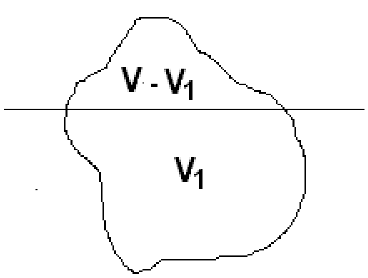
$$F = (h_2 - h_1)\, gA$$

$$F = h \rho g A$$

But volme V = hA
Thus F = Vρg
∴ force of Buoyancy = Vρg = Weight of liquid displaced

Expression for immersed volume of a floating Body

Let a solid of volume V and density ρ floats in liquid of density ρ0. Volume V1 of the body is immersed inside the liquid The



Let a solid of volume V and density ρ floats in liquid of density ρ_0. Volume V_1 of the body is immersed inside the liquid The weight of floating body = Vρg
The weight of the displaced liquid = $V_1\rho 0g$
For the body to float
Weight of body = Weight of liquid displaced
$V\rho g = V_1 \rho 0 g$

$$V_1 / V = \rho / \rho_0$$

$$V_1 = \rho V / \rho_0$$

∴ Immersed volume = mass of solid / density of liquid
From above it is clear that density of the solid volume must be less than density of the liquid to enable it to float freely in the liquid. How ever a metal vessel floats in water though the density of metal is much higher that the that of eater because floating bodies are hollow inside and hence displaces large volume. When thy float on water, the weight of the displaced water is equal to the weight of the body

Laws of floatation

The principle of Archimedes may be applied to floating bodies to give the laws of flotation

(i) When a body floats freely in a liquid the weight of the floating body is equal to the weight of the liquid displaced

(ii) The centre of gravity of the displaced liquid B (called the centre of buoyancy) lies vertically above or below the centre of gravity of the floating body G

Solved numerical

Q) A stone of mass 0.3kg and relative density 2.5 is immersed in a liquid of relative density 1.2. Calculate the resultant up thrust exerted on the stone by the liquid and the weight of stone in liquid

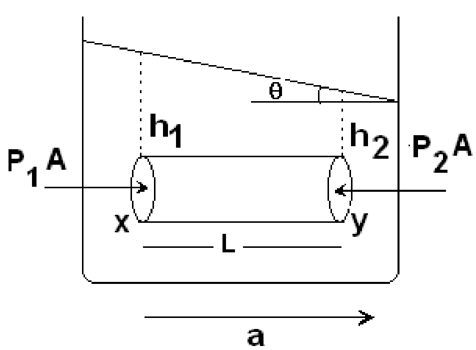
Solution

Volume of stone V = mass/density
V = 0.3/2.5 = 0.12 m3
Upward thrust =buoyant force = Vρ0g =0.12×1.2×9.8 = 1.41 N
Weight of stone in liquid = Gravitational force – buoyant force = 0.3×9.8 – 1.41 = 1.53 N or 0.156 kg wt

Liquid in accelerated Vessel

Variation of pressure and force of buoyancy in a liquid kept in accelerated vessel
Consider a liquid of density ρ kept in a vessel moving with acceleration **a in upward** direction. Let height of liquid column be h
Then effective gravitational acceleration on liquid = g +a
Thus pressured exerted at depth h $P = P_0 + \rho(g+a)\,h$
Similarly if liquid in container moves down with acceleration a
Then effective gravitational acceleration on liquid = g –a
Thus pressure exerted at depth h, $P = P_0 + \rho(g-a)\,h$ Also Buoyant force on immersed body when liquid is moving up
$F_B = V\rho(g+a)$
Buoyant force on immersed body when liquid is moving down
$F_B = V\rho(g+a)$
V is volume of the liquid displaced

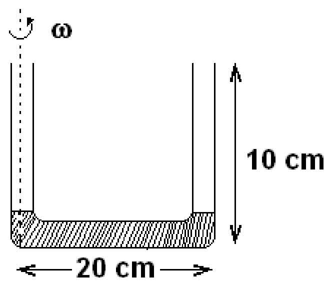
Shape of free surface of a liquid in horizontal accelerated vessel



When a vessel filled with liquid accelerates horizontally. We observe its free surface inclined at some angle with horizontal. To find angle θ made by free surface with horizontal, consider a horizontal liquid column including two points x and y at the depth of h_1 and h_2 from the inclined free surface of liquid as shown in figure
Force on area at x = $P_1A = h_1\rho g$
Pseudo force at y= mass of liquid tube of length L and cross sectional area A ×acceleration Pseudo force at y=ρ(LA)
Total force at y = P_1A + Pseudo force Force on area at y = $h_1\rho g + \rho(LA)$
Since liquid is in equilibrium Force on area at x = Force on area at y
$h_1\rho g = h_1\rho g + \rho(LA)$ $(h_1 - h_2)$ g = La From geometry of figure $h_1 - h_2/L$

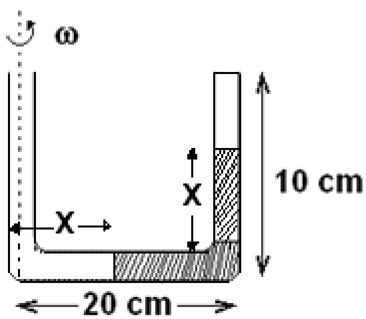
$= a/g$ $\tan\theta = a/g$

Q) Length of a horizontal arm of a U-tube is 20cm and end of both the vertical arms are



open to a pressure 1.01×10^3 N/m^2 . Water is poured into the tube such that liquid just fills horizontal part of the tube is then rotated about a vertical axis passing through the other vertical arm with angular velocity ω. If length of water in sealed tube rises to 5cm, calculate ω. Take density of water = 10^3 kg/m3 and g = 10 m/s^2 . Assume temperature to be constant.

Solution



when tube is rotated liquid will experience a centrifugal force thus water moves up in second arm of the U tube. When centrifugal force + pressure in first arm = force due to pressure in second closed arm +force due to liquid column then equilibrium condition is established ---eq(1)

Calculation of force due to pressure in closed tube Before closing pressure Pi = 1.01×10^3 N/m^2 Volume before closing Vi= 0.1A (A is area of cross-section) After closing the other arm Pressure Pf and volume Vf = 0.05A From equation PiVi = PfVf (1.01×10^3)× 0.1A = Pf ×(0.05A)
Pf = 2.02×10^3 Force due to pressure = (2.02×10^3)×A

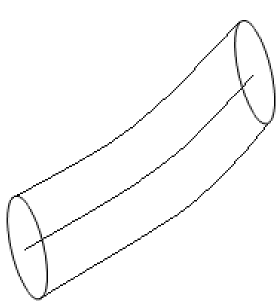
Pressure in first arm = 1.01×10^3

Calculation of force due to liquid column in second arm Height of liquid column = 0.05 m Thus pressure due to column = hpg = $0.05 \times 10^3 \times 10$ =500 N/m^2 Force due to liquid column PA= 0.5A

Calculation of centrifugal force Mass of the liquid in horizontal part = volume ×density = $(0.2-0.05)A \times 10^3$=150A Centre of mass of horizontal liquid from first arm 'r' =0.05 + 0.2−0.05 / 2 = 0.125 m Centrifugal force = mω2 r = $150A \times \omega^2 \times 0.125$ = $(18.75A)\omega^2$ Now substituting values in equation 1 we get $(18.75A)\times\omega^2$ +(1.01×10^3)×A =(2.02×10^3)×A + 500A $(18.75)\times\omega^2$ +1.01×10^3 =(2.02×10^3) + 500 ω = 8.97 rad/s

Fluid dynamics

Tube of flow



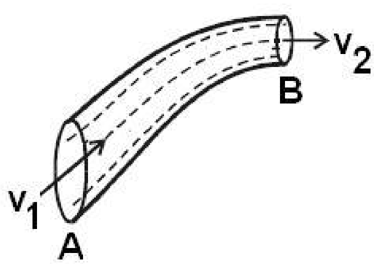
In a fluid having a steady flow, if we select a finite number of streamlines to form a bundle like the streamline pattern shown in the figure, the tubular region is called a tube of flow. The tube of flow is bounded by a streamlines so that by fluid can flow across the boundaries of the tube of flow and any fluid that enters at one end must leave at the other end.

Turbulent flow

When the velocity of a liquid exceeds the critical velocity, the path and velocities of the liquid become disorderly. At this stage, the flow loses all its orderliness and is called turbulent flow. Some examples of turbulent flow are :

(i) After rising a short distance, the smooth column of smoke from an incense stick breaks up into irregular and random patterns.

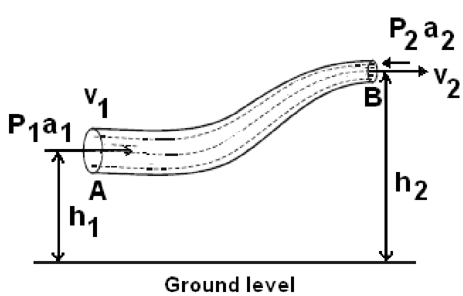
(ii) The flash - flood after a heavy rain. Critical velocity of a liquid can be defined as that velocity of liquid upto which the flow is streamlined and above which its flow becomes turbulent. Equation of continuity Consider a non-viscous liquid in streamline flow through a tube AB of varying cross section as shown in Fig. Let a1 and a2 be the area of cross section, v1 and v2 be the velocity of flow of the liquid at A and B respectively.



∴ Volume of liquid entering per second at A = a_1v_1. If ρ is the density of the liquid, then mass of liquid entering per second at A = $a_1v_1\rho$. Similarly, mass of liquid leaving per second at B = $a_2v_2\rho$ If there is no loss of liquid in the tube and the flow is steady, then mass of liquid entering per second at A = mass of liquid leaving per second at B (i.e) $a_1v_1\rho = a_2v_2\rho$ or $a_1v_1 = a_2v_2$ i.e. av = constant This is called as the equation of continuity. From this equation v is inversely proportional to area of cross-section along a tube of flow i.e. the larger the area of cross section the smaller will be the velocity of flow of liquid and vice-versa.

Bernoulli's Equation

The theorem states that the work done by all forces acting on a system is equal to the change in kinetic energy of the system



Consider streamline flow of a liquid of density ρ through a pipe AB of varying cross section. Let P_1 and P_2 be the pressures and a_1 and a_2, the cross sectional areas at A and B respectively. The liquid enters A normally with a velocity v_1 and leaves B normally with a velocity v_2. The liquid is accelerated against the force of gravity while flowing from A to B, because the height of B is greater than that of A from the ground level. Therefore P_1 is greater than P_2. This is maintained by an external force. The mass m of the liquid crossing per second through any section of the tube in accordance with the equation of continuity is $a_1v_1\rho$ = $a_2v_2\rho$ = m Or

$$a_1v_1 = a_2v_2 = m\,\rho$$

As $a_1 > a_2$, $v_1 < v_2$

The force acting on the liquid at A = P_1a_1
The force acting on the liquid at B = $P_2\,a_2$
Work done per second on the liquid at A = $P_1a_1 \times v_1 = P_1V$
Work done by the liquid at B = $P_2a_2 \times v_2 = P_2V$

∴ Net work done per second on the liquid by the pressure energy in moving the liquid from A to B is = P1V – P2V If the mass of the liquid flowing in one second from A to B is m, then increase in potential energy per second of liquid from A to

B is = $mgh_2 - mgh_1$
Increase in kinetic energy per second of the liquid.

$1/2\, mv^2{}_2 - 1/2\, mv^1{}_2$

According to work-energy principle, work done per second by the pressure energy = (Increase in potential energy + Increase in kinetic energy) per second

$P_1V - P_2V = (mgh_2 - mgh_1) + (1/2\, mv^2{}_2 - 1/2\, mv^1{}_2)$

$P_1V + mgh_1 + 1/2\, mv^1{}_2 = P_2V + mgh_2 + 1/2\, mv^2{}_2$

$P_1V/\, V + m/\, V\, gh_1 + 1/\, 2\, m\, /V\, v^1{}_2 = P_2V/V + m/V\, gh_2 + 1/2\, m/V\, v^2{}_2$

$P_1 + \rho gh_1 + 1/2\, \rho v^1{}_2 = P_2 + \rho gh_2 + 1\, 2\, \rho v^2{}_2$

Since subscripts 1 and 2 refer to any location on the pipeline, we can write in general

$P + \rho gh + 1\, 2\, mv^2 = constant$

The above equation is called Bernoulli's equation for steady non-viscous incompressible flow. Dividing the above equation by gh we can rewrite the above equation as

$$h + v^2/\, 2g + P\, \rho g = constnat, \text{ which is called total head}$$

Term h is called elevation head or gravitational head

$$v^2/2g \text{ is called velocity head}$$

$$P/\rho g \text{ is called pressure head}$$

Above equation indicates for **ideal liquid velocity increases when pressure decreases and vice-versa**

Q) A vertical tube of diameter 4mm at the bottom has a water passing through it. If the pressure be atmospheric at the bottom where the water emerges at the rate of 800gm per minute, what is the pressure at a point in the tube 5cm above the bottom where the diameter is 3mm

Solution

Rate of flow of water = 800 gm/min = (40/3)gm/sec
Now mass of water per sec = velocity ×area × density
40/3 = V1 ×[π (0.2) ^{2}]×1
V1 = (333.33/π) cm/sec
Now $A_1V_1 = A_2V_2$ Thus
$V_2 = (4/3)V_1$ $V_2 = (444.44/\pi)$ cm/sec is the velocity at height 25cm
Now P_1 = atmospheric pressure = 1.01×10^7 dyne
Now from Bernoulli's equation

$P_1 + \rho gh_1 + 1/2\, \rho v^1{}_2 = P_2 + \rho gh_2 + 1\, 2\, \rho v^2{}_2$

$1.01 \times 10^6 + 0 + 1\, 2\, (333.33\, \pi)^2 = P_2 + 1 \times 981 \times 25 + 1\, 2\, (444.44\, \pi)^2$

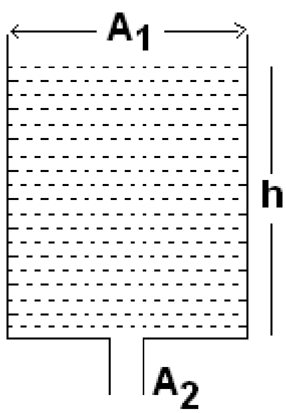
On solving $P_2 = 0.98\times10^6$ dyne
Now pressure = $h\rho_0 g$ here ρ_0 is density of mercury = 13.6 in cgs system

$0.98\times10^6 = h\times980\times13.6$

H = 73.5 cm of Hg

Speed of Efflux

As shown in figure a tank of cross-sectional area A, filled to a depth h with a liquid of density ρ . There is a hole of cross-section area A2 at the bottom and the liquid flows out



of the tank through the hole $A_2 << A_1$ Let v_1 and v_2 be the speeds of the liquid at A_1 and A_2. As both the cross sections are opened to the atmosphere, the pressure there equals to atmospheric pressure Po. If the height of the free surface above the hole is h[1] Bernoulli's equation gives

$$P_0 + 1/2\ \rho v^1{}_2 + \rho gh = P_0 + 1/2\ \rho v^2{}_2$$

By the equation of continuity, $A_1v_1 = A_2 v_2$

$$P_0 + 1/2\ \rho\ ({}_2/A_1)\ 2\ v^2{}_2 + \rho gh = P_0 + 1\ 2\ \rho v^2{}_2$$

$$[1 - ({}_2/A_1)^{\ 2}]\ v^2{}_2 = 2gh$$

$$v_2 = \surd\ 2gh\ /\ 1-(A_2 A_1)^2$$

If $A_2 <<< A_1$, the equation reduces to v2 =√(2gh) The speed of efflux is the same as the speed a body that would acquire in falling freely through a height h. This is known as Torricelli's theorem